THE HOMOLOGY OF HEISENBERG LIE ALGEBRAS OVER FIELDS OF CHARACTERISTIC TWO

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Abstract. The generating function of the Betti numbers of the Heisenberg Lie algebra over a field of characteristic 2 is calculated using discrete Morse theory.

The Heisenberg Lie algebra of dimension $2n+1$, denoted by $\mathfrak{h}_n$, is the vector space with basis $B = \{z, x_1, \dots, x_n, y_1, \dots y_n\}$ where the only non-zero Lie products of basis elements are

$$[x_i, y_i] = -[y_i, x_i] = z.$$

In this paper the Betti numbers of the homology of $\mathfrak{h}_n$ over a field of characteristic 2 is computed with the aid of algebraic discrete Morse theory from [Skö]. The notation from [Skö] will be freely used.

Theorem 1. *The generating function of the Betti numbers of the Heisenberg Lie algebra over a field of characteristic* 2 *is*

$$\sum_{i\geq 0} \dim_k H_i(\mathfrak{h}_n)\, t^i = \frac{(1+t^3)(1+t)^{2n} + (t+t^2)(2t)^n}{1+t^2}$$

When the ground field of $\mathfrak{h}_n$ has characteristic 0, Santharoubane [San83] has shown that

$$\dim_k H_i(\mathfrak{h}_n) = \binom{2n}{i} - \binom{2n}{i-2},$$

(the need for the ground field to have characteristic 0 is not explicitly mentioned).

Let us first recall the construction of the Chevalley–Eilenberg complex $\mathbf{V}$ of $\mathfrak{h}_n$, whose homology is the homology of $\mathfrak{h}_n$: the complex $\mathbf{V}$ is given by

$$0 \longrightarrow \bigwedge^{2n+1} \mathfrak{h}_n \longrightarrow \cdots \longrightarrow \bigwedge^{i} \mathfrak{h}_n \longrightarrow \cdots \longrightarrow \bigwedge^{2} \mathfrak{h}_n \longrightarrow \mathfrak{h}_n \longrightarrow 0$$

with the differential

$$\bar{d}(w_1 \wedge \cdots \wedge w_n) = \sum_{i<j} (-1)^{i+j} [w_i, w_j] \wedge w_1 \wedge \cdots \wedge \widehat{w_i} \wedge \cdots \wedge \widehat{w_j} \wedge \cdots \wedge w_n$$

for $w_i \in B$.

The p-th homology (with trivial coefficients) of $\mathfrak{h}_n$, can now be obtained as the p-th homology group of the complex $\mathbf{V}$.

If $I = \{i_1, \dots, i_s\}$ is a subset of $[n]$, we will use the notation x_I for the element $x_{i_1} \wedge \cdots \wedge x_{i_s}$, (and similarly for y_I).

Proof. The result is proved by constructing a Morse matching M on the digraph $G_{\mathbf{V}}$, and showing that when π is the projection coming from the splitting homotopy of M, we have that $\pi(\mathbf{V})$ has trivial differential.

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The decomposition of the Chevalley–Eilenberg complex we will use is the obvious; we consider the basis for $\mathbf{V}$ given by $\{z \wedge x_I \wedge y_J, x_I \wedge y_J \mid I, J \subseteq [n]\}$.

Let the matching M consist of the following edges in $G_{\mathbf{V}}$:

$$x_I \wedge y_J \to z \wedge x_{I \setminus \{k\}} \wedge y_{J \setminus \{k\}}$$

whenever $\max(I^c \cap J^c) < \max(I \cap J)$ and $k = \max(I \cap J)$.

There are now two kinds of unmatched elements: first the elements $z \wedge x_I \wedge y_J$, with $\max(I^c \cap J^c) < \max(I \cap J)$, and then the elements $x_I \wedge y_J$, with $\max(I^c \cap J^c) > \max(I \cap J)$.

When $x_I \wedge y_J \in M^+$, there is exactly one element $z \wedge x_{I'} \wedge y_{J'}$ with $x_I \wedge y_J \to z \wedge x_{I'} \wedge y_{J'}$ that is not in M^0, which implies that there can be no directed cycle in the graph $G_{\mathbf{V}}^M$. Together with the fact that for all edges in $G_{\mathbf{V}}$ the corresponding component of the differential is an isomorphism, this implies that M is a Morse matching.

We will now see that the differential in $\pi(\mathbf{V})$ is zero. For an element $z \wedge x_I \wedge y_J \in M^0$ it is obvious that $d\pi(z \wedge x_I \wedge y_J) = \pi d(z \wedge x_I \wedge y_J) = 0$. For $x_I \wedge y_J \in M^0$ with $m = \max(I^c \cap J^c)$ we get that

$$\pi(x_I \wedge y_J) = x_I \wedge y_J + \sum_{i \in I \cap J} x_{(I \setminus \{i\}) \cup \{m\}} \wedge y_{(J \setminus \{i\}) \cup \{m\}},$$

from which it is easily seen that $d\pi(x_I \wedge y_J) = 0$. From [Skö, Theorem 1] now follows that the i-th Betti number is equal to the number of unmatched vertices in homological degree i.

For the computation of the generating function we introduce the elements $u_i = x_i \wedge y_i$, and we begin by counting the critical vertices $z \wedge x_I \wedge y_J \wedge u_K$ and $x_I \wedge y_J \wedge u_K$ when $I \cup J = L$ for a fixed set $L \subseteq [n]$.

If $L = [n]$, the critical vertices are all $z \wedge x_I \wedge y_J$ and $x_I \wedge y_J$ and they contribute with $(1+t)(2t)^n$ toward the homology.

If $L \neq [n]$, then the critical vertices of the form $z \wedge x_I \wedge y_J \wedge u_K$ are those with $\max([n] \setminus (I \cup J)) \in K$ so they contribute with $t^3(2t)^{|L|}(1+t^2)^{n-|L|-1}$ toward the homology. The critical vertices of the form $x_I \wedge y_J \wedge u_K$ are those with $\max([n] \setminus (I \cup J)) \notin K$ and thus contribute with $(2t)^{|L|}(1+t^2)^{n-|L|-1}$ toward the homology.

Summing up we get

$$\begin{aligned}
f(t) &= (1+t)(2t)^n + (1+t^3) \sum_{L \subset [n]} (2t)^{|L|}(1+t^2)^{n-|L|-1} \\
&= (1+t)(2t)^n + (1+t^3) \sum_{i=0}^{n-1} \binom{n}{i} (2t)^i (1+t^2)^{n-i-1} \\
&= (1+t)(2t)^n + (1+t^3)(1+t^2)^{-1}((1+2t+t^2)^n - (2t)^n) \\
&= \frac{(1+t)(1+t^2)(2t)^n}{1+t^2} + \frac{(1+t^3)(1+t)^{2n} - (1+t^3)(2t)^n}{1+t^2} \\
&= \frac{(1+t^3)(1+t)^{2n} + (t+t^2)(2t)^n}{1+t^2}
\end{aligned}$$

□

References

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